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On topological BE-algebras

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Abstract. In this paper, we study some properties of uniform topology and topological BE-algebras and compare this topologies.

1. Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki as a generalization of the concept of set-theoretic difference and propositional calculus([3],[4]). In [9], J. Neggeres and H. S. Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [6] introduced a new notion, called BH-algebras, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of BE-algebras was introduced by H. S. Kim and Y. H. Kim [7].
In section 3 we study some properties of uniform topology. In section 4 we study some general properties of topological BE-algebras, and finally in section 5 we obtain some relationships between this topologies.

2. preliminaries

Recall that a set X with a family $\tau = \{U\}$ of its subsets is called a *topological space*, denoted by (X, τ), if $X, \emptyset \in \tau$, the intersection of any finite number of members of τ is in τ and the arbitrary union of members of τ is in τ. The members of τ is called *open* sets of X. The complement $X \setminus U$ of an open set U is said to be *closed* set. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_\tau B$). A subset P of X is said to be a *neighborhood* of $x \in X$, if there exists an open set U such that $x \in U \subseteq P$.
A subfamily $\{U_\alpha\}$ of τ is said to be a *base* of τ if for each $x \in U \in \tau$ there exists an α such that $x \in U_\alpha \subseteq U$, or equivalently, each U in τ is the union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of τ is said to form a *subbase* for τ if the family of finite intersections of members of $\{U_\beta\}$ forms a base of τ.

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Let (X, τ) be a topological space. We have the following separation axioms in (X, τ):

- $\mathbf{T_0}$: For each $x, y \in X$ and $x \neq y$, there is at least one in an open neighborhood excluding the other.
- $\mathbf{T_1}$: For each $x, y \in X$ and $x \neq y$, each has an open neighborhood not containing the other.
- $\mathbf{T_2}$: For each $x, y \in X$ and $x \neq y$, both have disjoint open neighborhoods U, V such that $x \in U$ and $y \in V$.

A topological space satisfying T_i is called a *T_i-space*. A T_2-space is also known as a *Hausdorff space*.

Definition 2.1. Let $(X, *)$ be an algebra of type 2 and τ be a topology on A. Then $\mathcal{X} = (X, *, \tau)$ is called a

- (i) *Left (right) topological algebra*, if for all a in X the map $X \hookrightarrow X$ is defined by $x \hookrightarrow a * x$ ($x \hookrightarrow x * a$) is continuous, or equivalently, if for any x in X and any open set U of $a * x$ $(x * a)$, there exists an open set V of x such that $a * V \subseteq U$ ($V * a \subseteq U$).
- (ii) *Semitopological algebra*, or operation $*$ is separately continuous, if X is right and left topological algebra.
- (iii) *Topological algebra*, if the operation $*$ is continuous, or equivalently, if for any x, y in X and any open set(neighborhood) W of $x * y$ there exist two open sets(neighborhoods) U and V of x and y, respectively, such that $U * V \in W$.

Let X be a nonempty set and U, V be any subsets of $X \times X$. Define
$U \circ V = \{(x, y) \in X \times X : (x, z) \in U \text{ and } (z, y) \in V, \text{ for some } z \in X\}$,
$U^{-1} = \{(y, x) : (x, y) \in U\}$, $\triangle = \{(x, x) : x \in X\}$.

Definition 2.2 ([5])**.** By an uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

- (i) $\triangle \subseteq U$, for any $U \in \mathcal{K}$,
- (ii) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
- (iii) if $U \in \mathcal{K}$, then there exist $V \in \mathcal{K}$ such that $V \circ V \subseteq U$,
- (iv) if $U, V \in \mathcal{K}$ then $U \cap V \in \mathcal{K}$,
- (v) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a uniform structure (uniform space).

Let $x \in X$ and $U \in \mathcal{K}$. Define $U[x] = \{y \in X : (x, y) \in U\}$.

Definition 2.3 ([11])**.** A BE-algebra is an algebra $(X, *, 1)$ of type $(2, 0)$ such that satisfying the following axioms:

1. $x * x = 1$ for all $x \in X$,
2. $x * 1 = 1$ for all $x \in X$,
3. $1 * x = x$ for all $x \in X$,
4. $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

A relation $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 1$. If X is a BE-algebra and $x, y \in X$, then $x * (y * x) = 1$.

Definition 2.4 ([11])**.** We say that a BE-algebra X is commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

Proposition 2.1 ([1])**.** *Let X be a commutative BE-algebra and $x, y, z \in X$. Then,*

(5) $x * y = y * x = 1 \Rightarrow x = y$,
(6) $(x * y) * ((y * z) * (x * z)) = 1$.

Definition 2.5 ([11])**.** We say that a BE-algebra X is transitive if $(y * z) \leq (x * y) * (x * z)$ for all $x, y, z \in X$.

Proposition 2.2 ([11])**.** *If X is a commutative BE-algebra, then it is transitive.*

Definition 2.6 ([11])**.** Let A be a BE-algebra. A filter is a nonempty set $F \subseteq X$ such that for all $x, y \in A$

(i) $1 \in F$,
(ii) $x \in F$ and $x * y \in F$ imply $y \in F$.

Let F be a filter in X. If $x \in F$ and $x \leq y$ then $y \in F$.

Definition 2.7 ([11])**.** A filter F of a BE-algebra X is said to be normal if it satisfies the following condition:

$$x * y \in F \Rightarrow [(z * x) * (z * y) \in F \text{ and } (y * z) * (x * z) \in F]$$

for all $x, y, z \in X$.

Proposition 2.3 ([11])**.** *If X is a transitive BE-algebra, then every filter of X is normal.*

Proposition 2.4 ([11])**.** *Let F be a normal filter of a BE-algebra X. Define*

$$x \equiv^F y \ \Leftrightarrow \ x * y \ , \ y * x \in F.$$

Then

(i) *$\equiv^F$ is a congruence relation on X, i.e., it is a equivalence relation on X such that for each $a, b, c, d \in X$ if $a \equiv^F b$ and $c \equiv^F d$, then $a * c \equiv^F b * d$.*
(ii) *Let $\frac{x}{F} = \{y \in x : x \equiv^F y\}$ be an equivalence class of x and $\frac{X}{F} = \{\frac{x}{F} : x \in X\}$. Then $\frac{X}{F}$ is a BE-algebra under the binary operations given by:*

$$\frac{x}{F} * \frac{y}{F} = \frac{x * y}{F}.$$

Definition 2.8 ([2])**.** Let X be a BE-algebra. If there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$, then X is called a bounded BE-algebra.

Notation. From now, in this paper $(X, *, 1)$ is a commutative BE-algebra.

3. Uniform topology on BE-algebras

Theorem 3.1 ([8]). *Let Λ be an arbitrary family of filters of a BE-algebra X which is closed under intersection. If $U_F = \{(x,y) \in X \times X : x \equiv^F y\}$ and $\mathcal{K}^* = \{U_F : F \in \Lambda\}$, then $\mathcal{K}^*$ satisfies in the conditions $(i) \sim (iv)$ of Definition 2.2.*

Theorem 3.2 ([8]). *Let $\mathcal{K} = \{U \subseteq X \times X : U_F \subseteq U,$ for some $U_F \in \mathcal{K}^*\}$. Then the pair $(X, \mathcal{K})$ is an uniform structure.*

Theorem 3.3 ([8]). *Given a BE-algebra X, then*

$$\tau = \{G \in X : \forall x \in G\ \exists U \in \mathcal{K}\ s.t.\ U[x] \subseteq G\}$$

is a topology on X.

Definition 3.1. Let $(X, \mathcal{K})$ be an uniform structure. Then the topology τ is called an uniform topology on X induced by $\mathcal{K}$.

We denote the uniform topology obtained by a family Λ, by τ_Λ and if $\Lambda = \{F\}$, then we denote it by τ_F.

Note that for any $x \in X$, $U[x]$ is an open neighborhood of x.

Theorem 3.4 ([8]). *The pair (X, τ_Λ) is a topological BE-algebra.*

Notation. Let Λ be a family of filters of a BE-algebra X which is closed under intersection and $F \in \Lambda$ and $A \subseteq X$. Then we define $U_F[A] = \bigcup_{a \in A} U_F[a]$.

Theorem 3.5. *Let Λ be a family of filters of a BE-algebra X which is closed under intersection and $F \in \Lambda$ and $A \subseteq X$. Then the closure of A is $\bigcap\{U_F[A] : U_F \in \mathcal{K}^*\}$ and it is denoted by $\overline{A}$ in the topological space (X, τ_Λ).*

Proof. Let $x \in \overline{A}$. Then $U_F[x]$ is an open neighborhood of x and we have $U_F[x] \cap A \neq \emptyset$, for all $F \in \Lambda$. Hence there exists $y \in A$ such that $y \in U_F[x]$. Hence $(x, y) \in U_F$ for all $F \in \Lambda$. Thus $x \in U_F[y] \subseteq U_F[A]$ for all $F \in \Lambda$. Conversely, let $x \in U_F[A]$ for all $F \in \Lambda$. Then there exists $y \in A$ such that $x \in U_F[y]$ and so $U_F[x] \cap A \neq \emptyset$ for all $F \in \Lambda$. Therefore $x \in \overline{A}$. □

Theorem 3.6. *Let Λ be a family of filters of a BE-algebra X which is closed under intersection, K be a compact subset of X and W be an open set containing K. Then $K \subseteq U_F[K] \subseteq W$.*

Proof. Since W is an open set containing K, for each $k \in K$ we have $U_{F_k}[k] \subseteq W$ for some $F_k \in \Lambda$. Hence $K \subseteq \bigcup_{k \in K} U_{F_k}[k] \subseteq W$. Since K is a compact subset of X, there exist $k_1, k_2, ..., k_n$ such that

$$K \subseteq U_{F_{k_1}}[k_1] \cup U_{F_{k_2}}[k_2] \cup \cdots \cup U_{F_{k_n}}[k_n].$$

Put $F = \bigcap_{i=1}^n F_{k_i}$. We claim that $U_F[K] \subseteq W$ for each $k \in K$. Let $k \in K$. Then there exists $1 \leq i \leq n$ such that $k \in U_{F_{K_i}}[K_i]$ and hence $k \equiv^{F_{K_i}} k_i$. Now, let $y \in U_F[k]$, then $y \equiv^F k$. Therefore we have $y \equiv^{F_{K_i}} k_i$ and hence $y \in U_{F_{K_i}}[K_i] \subseteq W$. Hence $U_F[k] \subseteq W$ for any $k \in K$. Thus $K \subseteq U_F[K] \subseteq W$. □

Theorem 3.7. *Let Λ be a family of filters of a BE-algebra X which is closed under intersection, K be a compact subset of X and C be a closed subset of X. If $K \cap C = \emptyset$, then $U_F[K] \cap U_F[C] = \emptyset$ for some $F \in \Lambda$.*

Proof. Since $K \cap C = \emptyset$ and C is closed, $X \setminus C$ is an open set containing K. By Theorem 3.6 there exists $F \in \Lambda$ such that $U_F[K] \subseteq X \setminus C$. If $U_F[K] \cap U_F[C] \neq \emptyset$, then there exists $y \in X$ such that $y \in U_F[k]$ and $y \in U_F[c]$ for some $k \in K$ and $c \in C$, respectively. Hence $k \equiv^F c$ and then $c \in U_F[k] \subseteq U_F[K]$. This contradicts to the fact that $U_F[K] \subseteq X \setminus C$. Hence $U_F[K] \cap U_F[C] = \emptyset$. □

4. Topological BE-algebras

Theorem 4.1. *Let $\mathcal{F}$ be a family of filters in a BE-algebra X such that for each $F_1, F_2 \in \mathcal{F}$, there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. Then there is a topology τ on X such that $(X, *, \tau)$ is a topological BE-algebra.*

Proof. Define $\tau = \{U \subseteq X : \forall x \in U\ \exists F \in \mathcal{F}\ s.t.\ x/F \subseteq U\}$. For each $x \in X$ and $F \in \mathcal{F}$, the set $x/F \in \tau$, because if y is an arbitrary element of x/F then $y/F \subseteq x/F$. It is easy to see that τ is a topology on X. We prove that $*$ is continuous. For this, suppose $x * y \in U \in \tau$. Then for some $F \in \mathcal{F}$, $\frac{x*y}{F} \subseteq U$. Now, x/F and y/F are two open neighborhoods of x and y, respectively, such that $x/F * y/F \subseteq \frac{x*y}{F} \subseteq U$. □

Example 4.1. Let $X = \{a, b, c, d, 1\}$. Define a binary operation $*$ on X as follow:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Easily we can check that $(X, *, 1)$ is a BE-algebra [9]. Let $\tau = \{\{1, a, c\}, \{b, d\}, X, \emptyset\}$. Then $(X, *, \tau)$ is a topological BE-algebra [8].

Theorem 4.2. *Let $(X, *, \tau)$ be a topological BE-algebra.*

(*i*) *(X, τ) is discrete if and only if $\{1\}$ is open.*
(*ii*) *(X, τ) is Hausdorff if and only if $\{1\}$ is closed.*

Proof. (*i*) Let $\{1\}$ be an open subset of X. Then by (1), $x * x = 1 \in \{1\}$ for all $x \in X$. Since $(X, *, \tau)$ is a topological BE-algebra, there exist neighborhoods U and V of x such that $U * V \subseteq \{1\}$. Put $W = U \cap V$. Then $1 = x * x \in W * W \subseteq U * V \subseteq \{1\}$ and so $W * W = \{1\}$. We claim that $W = \{x\}$. Let $y \in W$. Then $x * y \in W * W = \{1\}$ and $y * x \in W * W = \{1\}$. Hence $x = y$ and so $W = \{x\}$. The converse is trivial.
(*ii*) Suppose that $(X, *, \tau)$ is a Hausdorff space. We show that $X \setminus \{1\}$ is an open subset of X. Let $x \in X \setminus \{1\}$. Then $x \neq 1$. Hence there exist

neighborhoods U of x and V of 1 such that $U \cap V = \emptyset$. Thus $1 \notin U$. Therefore $x \in U \subseteq X \setminus \{1\}$ and so $X \setminus \{1\}$ is an open subset of X.
Conversely, let $\{1\}$ be closed and $x, y \in X$ such that $x \neq y$. Then $x*y \neq 1$ or $y*x \neq 1$. Let $x*y \neq 1$. Then $x*y \in X \setminus \{1\}$. Since $X \setminus \{1\}$ is open, there exist neighborhoods U of x and V of y such that $U * V \subseteq X \setminus \{1\}$. We claim that $U \cap V = \emptyset$. Let $U \cap V \neq \emptyset$. Let $y \in U \cap V$. Hence $1 = y*y \in U \cap V \subseteq X \setminus \{1\}$ which is a contradiction. Therefore (X, τ) is a Hausdorff space. □

Theorem 4.3. *Let $(X, *, \tau)$ be a topological BE-algebra. Then the following are equivalent:*

(*i*) *(X, τ) is a Hausdorff space,*
(*ii*) *(X, τ) is T_1,*
(*iii*) *(X, τ) is T_0.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (i)$ Let $x, y \in X$ and $x \neq y$. Then $x*y \neq 1$ or $y*x \neq 1$. Let $x*y \neq 1$. Since X is a T_0 space, there is an neighborhood U of $x*y$ such that $1 \notin U$. Since $(X, *, \tau)$ is a topological BE-algebra, there exist neighborhoods V of x and W of y such that $V*W \subseteq U$. We claim that $V \cap W = \emptyset$. Let $V \cap W \neq \emptyset$. Let $z \in V \cap W$. Hence $1 = z*z \in V*W \subseteq U$. This is a contradiction. Hence (X, τ) is a Hausdorff space. □

Theorem 4.4. *Let $(X, *, \tau)$ be a topological BE-algebra and F be a filter of X. Then 1 is an interior point of F if and only if F is open.*

Proof. Suppose that 1 is an interior point of F. Then there exists a neighborhood U of 1 such that $U \subseteq F$. Let $x \in F$ be an arbitrary element. Since $x * x = 1$, there exist neighborhoods V, W of x such that $V * W \subseteq U \subseteq F$. Now, for each $y \in W$, we have $x * y \in F$. Since F is a filter and $x \in F$, we have $y \in F$. Hence $W \subseteq F$ and so F is open. The converse is trivial. □

Theorem 4.5. *Let $(X, *, \tau)$ be a topological BE-algebra and F be a filter of X. If F is open, then F is closed.*

Proof. Let F be a filter of X which is open in (X, τ). We show that $X \setminus F$ is open. Let $x \in X \setminus F$. Since F is open, by Theorem 4.4, 1 is an interior point of F. Hence there exists a neighborhood U of 1 such that $U \subseteq F$. Since $x*x = 1$, there exist neighborhoods V and W of x such that $V*W \subseteq U \subseteq F$. We claim that $V \subseteq X \setminus F$. Let $V \nsubseteq X \setminus F$. Then there exists $y \in V \cap F$. For each $z \in W$, we have $y*z \in V*W \subseteq F$. Since $y \in F$ and F is a filter, $z \in F$. Hence $W \subseteq F$ and so $x \in F$ which is a contradiction. □

In Theorem 4.9 we will prove that the converse of Theorem 4.5 is also true.

Theorem 4.6. *Let $(X, *, \tau)$ be a topological BE-algebra. If $1 \in \bigcap_{U \in \tau} U$, then $B \subseteq X$ is open if and only if 1 is an interior point of B.*

Proof. If B is open, clearly, 1 is an interior point of B. Let 1 be an interior point of B and $x * x = 1$, there is an open neighborhood V of 1 such that $x * x = 1 \in V \subseteq B$. Since $*$ is continuous, there exists an open set W containing x such that $W * W \subseteq V$. By hypothesis, $1 \in W$, and hence $x \in W \subseteq W * W \subseteq V \subseteq B$. This proves that x is an interior point of B. □

Theorem 4.7. *Let $(X, *, \tau)$ be a topological BE-algebra and F_1 the least open set containing* 1. *If $x \in F_1$, then F_1 is the least open set containing x.*

Proof. Let $x \in F_1$ and U be an open set such that $x \in U$. Since $1*x = x \in U$, there exist open neighborhoods V of 1 and W of x such that $V * W \subseteq U$. We have $1 = x * x \in F_1 * W \subseteq V * W \subseteq U$. Therefore $1 \in U$. Since F_1 is the least open set such that $1 \in F_1$, $F_1 \subseteq U$. □

Theorem 4.8. *Let $(X, *, \tau)$ be a topological BE-algebra and F_1 the least open set containing* 1. *Then F_1 is a filter of X.*

Proof. Let $x, x * y \in F_1$. By Theorem 4.7, F_1 is the least open set containing x. Since $x * y \in F_1$, there exist open neighborhoods U of x and V of y such that $U * V \subseteq F_1$. Hence $y = 1 * y \in F_1 * V \subseteq U * V \subseteq F_1$ and therefore $y \in F_1$. □

Theorem 4.9. *Let $(X, *, \tau)$ be a topological BE-algebra and F a filter in X. If F is closed then F is open.*

Proof. Suppose that F is closed filter but not open. By Theorem 4.4, 1 is not an interior point of F. Hence $F_1 \nsubseteq F$, where F_1 is the least open set such that $1 \in F_1$. If $(X \setminus F) \cap F_1 = \emptyset$, then $F_1 \subseteq F$. Hence $(X \setminus F) \cap F_1 \neq \emptyset$. Since $(X \setminus F) \cap F_1$ is open, by Theorem 4.7, $(X \setminus F) \cap F_1 = F_1$. Thus $F_1 \subseteq X \setminus F$ and so $1 \in X \setminus F$ which is a contradiction. □

5. Comparsion τ and τ_Λ

In this section we assume that $(X, *, \tau)$ is a topological BE-algebra and $1 \neq x \in X$. The least open set containing x is denoted by U_x.

Lemma 5.1. *If $x * y \notin F_1$, then $y \notin U_x$ and $x \notin U_y$.*

Proof. Let $y \in U_x$. Then $\{x, y\} \subseteq U_x$. Since $x * y \in U_{x*y}$, there exist open neighborhoods V_1 of x and V_2 of y such that $V_1 * V_2 \subseteq U_{x*y}$. We have $y \in U_x \subseteq V_1$, $y \in U_y \subseteq V_2$ and then $1 = y * y \in U_x * U_y \subseteq U_{x*y}$. Put $z = x * y$. Since $z * z = 1 \in F_1$, there exist open neighborhoods W_1, W_2 of z such $W_1 * W_2 \subseteq F_1$. Then $1*z \in U_z * U_z \subseteq W_1 * W_2 \subseteq F_1$. Hence $x*y = z \in F_1$ which is a contradiction. Similarly, we can show that $x \notin U_y$. □

Theorem 5.1. *Let $(X, *, \tau)$ be a topological BE-algebra and τ_{F_1} be the uniform topology induced by filter F_1. Then τ is finer than τ_{F_1}.*

Proof. We will show that $U_{F_1}[x] = \bigcup_{y\in F_1[x]} U_y$ for all $x \in X$. Let $y \in U_{F_1[x]}$ and $z \in U_y$. If $z * y \notin F_1$ or $y * z \notin F_1$, then by Lemma 5.1, $z \notin U_y$. Thus $z * y \in F_1$ and $y * z \in F_1$. By (6), $(x * y) * ((y * z) * (x * z)) = 1 \in F_1$. Since $x * y \in F_1$, we have $(y * z) * (x * z) \in F_1$ and so $x * z \in F_1$ because $y * z \in F_1$. Similarly, we can show that $z * x \in F_1$. Hence $z \in U_{F_1}[x]$. Therefore $U_y \subseteq U_{F_1}[x]$ for all $y \in U_{F_1}[x]$ and so $\bigcup_{y\in F_1[x]} U_y \subseteq U_{F_1}[x]$. It is clear that $U_{F_1}[x] \subseteq \bigcup_{y\in F_1[x]} U_y$. □

Theorem 5.2. *Let $(X, *, \tau)$ be a topological BE-algebra and τ_{F_1} be a uniform topology induced by filter F_1. If there exists $U \in \tau$ such that $U \notin \tau_{F_1}$, then there exist $x \in U$ and $y \in U_{F_1}[x]$ such that $y \notin U$ and the following properties holds:*

(*i*) *$x, y \notin F_1$.*
(*ii*) *$a * y \notin U_{F_1}[x] \cap U$, for all $a \in F_1$.*
(*iii*) *If $d \in U_{F_1}[x] \cap U$, then $a * d \neq y$, for all $a \in F_1$.*

Proof. (*i*) If $x \in F_1$, then by Theorem 4.7, $F_1 \subseteq U$. Since $x \in F_1$, $y \in U_{F_1}[x]$ and F_1 is a filter, we have $y \in F_1 \subseteq U$ which is a contradiction. Let $y \in F_1$. Since $y \in U_{F_1}[x]$ and F_1 is a filter, then $x \in F_1$ which is a contradiction.
(*ii*) Suppose that there exists some $a \in F_1$ such that $a*y \in U_{F_1}[x]\cap U$. There exist open neighborhoods V_1 of a and V_2 of y such that $V_1 * V_2 \subseteq U_{F_1}[x]\cap U$. By Theorem 4.7, $F_1 \subseteq V_1$. Then $y = 1 * y \in F_1 * V_2 \subseteq U_{F_1}[x] \cap U$. Hence $y \in U_{F_1}[x] \cap U$ which is a contradiction.
(*iii*) Suppose that there exists $a \in F_1$ such that $a * d = y$ for some $d \in U_{F_1}[x] \cap U$. Since $1 * d = d \in U_{F_1}[x] \cap U$. there exist open neighborhoods V_1 of 1 and V_2 of d such that $V_1 * V_2 \subseteq U_{F_1}[x] \cap U$. Then $y = a * d \in F_1 * V2 \subseteq V_1 * V_2 \subseteq U_{F_1}[x] \cap U$. Hence $y \in U_{F_1}[x] \cap U$ which is a contradiction. □

Lemma 5.2. *Let $(X, *, \tau)$ be a topological BE-algebra and τ_{F_1} be the uniform topology induced by filter F_1. If $\tau_{F_1} \subsetneqq \tau$, then there exists $\emptyset \neq U \in \tau$ such that $U \subsetneqq U_{F_1}[x]$ for some $x \in X \setminus F_1$.*

Proof. If $\tau_{F_1} \subsetneqq \tau$, then there exists $V_1 \in \tau$ such that $V_1 \notin \tau_{F_1}$. By definition of uniform topology, there exists $x \in V_1$ such that $U_{F_1}[x] \nsubseteq V_1$. Hence $U_{F_1}[x] \cap V_1 \subsetneqq U_{F_1}[x]$. Put $U = U_{F_1}[x] \cap V_1$. Then $U \in \tau$ and $U \subsetneqq U_{F_1}[x]$. Suppose that $x \in F_1$. Then $U_{F_1}[x] = F_1$. Hence $x \in U$. By Theorem 4.7, $U_{F_1}[x] = F_1 \subseteq U$ which is a contradiction. □

Theorem 5.3. *Let $\{0, a, b, 1\}$ be a bounded BE-algebra and let $(X, *, \tau)$ be a topological bounded BE-algebra and τ_{F_1} be the uniform topology induced by filter F_1. Then $\tau = \tau_{F_1}$.*

Proof. **Case 1.** If $F_1 = \{1\}$ or $F_1 = X$, then it is clear that $\tau = \tau_{F_1}$.
Case 2. Suppose that $F_1 = \{x, 1\}$ where $x \in \{a, b\}$ and $\tau_{F_1} \subsetneqq \tau$. Without less of generality, we assume that $F_1 = \{a, 1\}$. By Lemma 5.3, there exists $U \in \tau$ such that $U \subsetneqq U_{F_1}[y]$ for some $y \in X \setminus F_1 = \{0, b\}$. If $U_{F_1}[y] = \{y\}$,

then $U = \emptyset$ which is a contradiction. So $U_{F_1}[y] = U_{F_1}[0] = U_{F_1}[b] = \{0, b\}$. Hence $b * 0 \in F_1$ and $0 * b \in F_1$. Then $b * 0 = a$. If $a * 0 \in F_1$, then $0 \in F_1$ and $F_1 = X$ which is a contradiction. Hence $a * 0 = b$. Therefore $U = \{0\}$ or $U = \{b\}$. Consider the following cases:

(1) Suppose that $U = \{0\}$. Since $1 * 0 = 0 \in U$, there exist $V, W \in \tau$ such that $1 \in V, 0 \in W$ and $V * W \subseteq U$. So

$$\{0, b\} = \{1 * 0, a * 0\} \subseteq F_1 * U \subseteq V * W \subseteq U,$$

which is a contradiction.

(2) Suppose that $U = \{b\}$. Since $a * 0 = b \in U$, there exist $V, W \in \tau$ such that $a \in V, 0 \in W$ and $V * W \subseteq U$. By Theorem 4.7, $F_1 \subseteq V$ and hence

$$\{0, b\} = \{1 * 0, a * 0\} \subseteq F_1 * W \subseteq V * W \subseteq U,$$

which is a contradiction.

Case 3. Suppose that $F_1 = \{a, b, 1\}$ and $\tau_{F_1} \subseteq \tau$. By Lemma 5.2, there exists $U \in \tau$ such that $\emptyset \neq U \subsetneqq U_{F_1}[y]$ for some $y \in X \setminus F_1$. Therefore $U \subsetneqq U_{F_1}[0] = \{0\}$ which is a contradiction.

Hence $\tau_{F_1} = \tau$ for all cases. □

Lemma 5.3. *Let $X = \{0, x, y, z, 1\}$ be a bounded BE-algebra and $F = \{x, 1\}$ be a filter of X. Then $y * z \neq x$ or $z * y \neq x$.*

Proof. Let $y * z = z * y = x$. Then $x * (y * z) = x * (z * y) = x * x = 1$. Consider following cases:

(1) Suppose that $x * y = 0$. Since $1 = x * (z * y) = z * (x * y)$, we get $z * 0 = 1$ and so $z \leq 0$. Since $0 \leq z$, we have $z = 0$ which is a contradiction.

(2) Suppose that $x * y = y$. Since $1 = x * (z * y) = z * (x * y)$, we have $z * y = 1$ which is a contradiction.

(3) Suppose that $x * y = z$. Since $y * (x * y) = 1$, we have $y * z = 1$ which is contradiction.

(4) If $x * y = x$ or $x * y = 1$, then $y \in F$ which is a contradiction. □

Lemma 5.4. *Let $X = \{0, x, y, z, 1\}$ be a bounded BE-algebra and $F = \{x, 1\}$ be a filter of X. Then*

(*i*) *if $U_F[y] = \{0, y\}$, then $x * 0 = y$,*

(*ii*) *if $U_F[y] = \{y, z\}$, then $x * y = z$ or $x * z = y$,*

(*iii*) *if $U_F[y] = \{0, y, z\}$, then ($x * y = z$ and $x * 0 = z$) or ($x * z = y$ and $x * 0 = y$).*

Proof. (*i*) Let $U_F[y] = \{0, y\}$, where $y \neq 0$. Then $y * 0 \in F$ and $0 * y \in F$. Therefore $y * x = x$. If $x * 0 \in F$, then $0 \in F$ which is a contradiction. Hence $x * 0 = z$ or $x * 0 = y$. Now, let $x * 0 = z$. Then $z * (x * 0) = z * z = 1$ and hence $x*(z*0)=1$. Therefore $x\leq z*0$. Since F is a filter, $z*0\in F$. Also $0*z=1\in F$. Hence

$$(z*0)*((0*y)*(z*y))=1,$$
$$(y*0)*((0*z)*(y*z))=1.$$

Therefore $z*y\in F$ and $y*z\in F$ and so $z\in U_F[y]$ which is a contradiction. Hence $x*0=y$.
(*ii*) By Lemma 5.3, $y*z=1, z*y=x$ or $y*z=x, z*y=1$. Let $y*z=1, z*y=x$. Hence $y\leq z$ and $z\leq x*y$ because $x*(z*y)=x*x=1$ hence $z*(x*y)=1$ and so $z\leq x*y$. If $x*y\in F$, then $y\in F$ which is a contradiction. Since $0\leq y\leq z\leq x*y$ and $x*y\notin F$, then $x*y=z$. Similarly, if $y*z=x, z*y=1$, then $x*z=y$.
(*iii*) Let $U_F[y]=\{0,y,z\}$. Then $y*0=x, z*0=x, y*z\in F$ and $z*y\in F$. By Lemma 5.3, $y*z=1, z*y=x$ or $y*z=x, z*y=1$. If $y*z=1$ and $z*y=x$, then $x*y=z$ similar to part (2). Since $y\leq z\leq x*0$ and $x*0\notin F$, then $x*0=z$. If $y*z=x$ and $z*y=1$, then $x*z=y$ similar to part (2). Since $z\leq y\leq x*0$ and $x*0\notin F$, then $x*0=y$. □

Theorem 5.4. *Let X be a bounded BE-algebra where $|X|=5$. If $(X,*,\tau)$ is a topological BE-algebra and $F=\{x,1\}$ is a filter in X, then $\tau=\tau_F$.*

Proof. Suppose that $\tau\neq\tau_F$. By Lemma 5.2, there exists $U\in\tau$ such that $U\subsetneqq U_F[y]$ for some $y\in X\setminus F$. Consider the following cases:
Case (1). $U_F[y]=\{y\}$. Then $U=\emptyset$.
Case (2). $U_F[y]=\{0,y\}$, where $y\neq 0$. By Lemma 5.4 part (1), $x*0=y$. Since $U\subsetneqq U_F[y]$, then $U=\{0\}$ or $U=\{y\}$.

(1) Suppose that $U=\{0\}$. Since $1*0=0\in U$, there exist $V,W\in\tau$ such that $1\in V, 0\in W$ and $V*W\subseteq U$. Then we have

$$\{0,y\}=\{1*0,x*0\}\subseteq F*U\subseteq V*W\subseteq U.$$

Hence $y\in U$, which is a contradiction.

(2) Suppose that $U=\{y\}$. Since $x*0=y$, then there exist $V,W\in\tau$ such that $x\in V, 0\in W$ and $V*W\subseteq U$. Then we have

$$\{0,y\}=\{1*0,x*0\}\subseteq F*W\subseteq V*W\subseteq U.$$

Hence $0\in U$, which is a contradiction.

Case (3). Suppose that $U_F[y]=\{y,z\}$, where $y,z\neq 0$. By Lemma 5.3, $y*z=1, z*y=x$ or $y*z=x, z*y=1$. Let $y*z=1, z*y=x$. Then by Lemma 5.4 part (2), $x*y=z$. Since $U\subsetneqq U_F[y]$, we have $U=\{y\}$ or $U=\{z\}$.

(1) Suppose that $U=\{y\}$. Since $1*y=y\in U$, there exist $V,W\in\tau$ such that $1\in V, y\in W$ and $V*W\subseteq U$.

$$\{y,z\}=\{1*y,x*y\}\subseteq F*U\subseteq V*W\subseteq U,$$

which is a contradiction.

(2) Suppose that $U = \{z\}$. Since $x * y = z \in U$, there exist $V, W \in \tau$ such that $x \in V, y \in W$ and $V * W \subseteq U$.

$$\{y, z\} = \{1 * y, x * y\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.

Case (4). Suppose that $U_F[y] = \{0, y, z\}$. By Lemma 5.3, $y*z = 1, z*y = x$ or $y * z = x, z * y = 1$. Let $y * z = 1$ and $z * y = x$. By Lemma 5.4 part (3), $x * y = z$ and $x * 0 = z$. Then

(1) Suppose that $U = \{0\}$. Since $1 * 0 = 0 \in U$, there exist $V, W \in \tau$ such that $1 \in V, 0 \in W$ and $V * W \subseteq U$.

$$\{0, z\} = \{1 * 0, x * 0\} \subseteq F * U \subseteq V * W \subseteq U = \{0\},$$

which is a contradiction.

(2) Suppose that $U = \{y\}$. Since $1 * y = y \in U$, there exist $V, W \in \tau$ such that $1 \in V, y \in W$ and $V * W \subseteq U$.

$$\{y, z\} = \{1 * y, x * y\} \subseteq F * U \subseteq V * W \subseteq U,$$

which is a contradiction.

(3) Suppose that $U = \{z\}$. Since $x * y = z \in U$, there exist $V, W \in \tau$ such that $x \in V, y \in W$ and $V * W \subseteq U$.

$$\{y, z\} = \{1 * y, x * y\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.

(4) Suppose that $U = \{0, z\}$. Since $x * y = z \in U$, there exist $V, W \in \tau$ such that $x \in V, z \in W$ and $V * W \subseteq U$.

$$\{y, z\} = \{1 * y, x * y\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.

(5) Suppose that $U = \{0, y\}$. Since $1 * y = y \in U$, there exist $V, W \in \tau$ such that $1 \in V, y \in W$ and $V * W \subseteq U$.

$$\{y, z\} = \{1 * y, x * y\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.

(6) Suppose that $U = \{y, z\}$. Since $x * 0 = z \in U$, then there exist $V, W \in \tau$ such that $x \in V, 0 \in W$ and $V * W \subseteq U$.

$$\{0, z\} = \{1 * 0, x * 0\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.

Hence $\tau = \tau_F$. □

Theorem 5.5. *Let X be a bounded BE-algebra where $|X| = 5$. If $(X, *, \tau)$ is a topological BE-algebra, then $\tau = \tau_{F_1}$.*

Proof. **Case (1).** If $F_1 = \{1\}$ or $F_1 = X$, then it is clear $\tau = \tau_{F_1}$.
Case (2). If $F_1 = \{x, 1\}$, then $\tau = \tau_{F_1}$ by Theorem 5.4.
Case (3). Suppose that $F_1 = \{z, x, 1\}$ but $\tau \neq \tau_{F_1}$. By Lemma 5.2, there exists $U \in \tau$ such that $U \subsetneqq U_{F_1}[a]$ for some $a \in X \setminus F_1 = \{0, y\}$. Then

(*i*) If $U_{F_1}[a] = \{a\}$, then $U = \emptyset$.
(*ii*) If $U_{F_1}[a] = U_{F_1}[y] = \{0, y\}$, then $y \leq x * 0$. Since $x * 0 \notin F_1$, thus $x * 0 = y$. Since $U \subsetneqq U_{F_1}$, we have $U = \{y\}$ or $U = \{0\}$.
(1) Suppose that $U = \{y\}$. Since $x * 0 = y \in U$, there exist $V, W \in \tau$ such that $x \in V, 0 \in W$ and $V * W \subseteq U$.

$$\{0, y\} \subseteq \{1 * 0, x * 0\} \subseteq F * W \subseteq V * W \subseteq U,$$

which is a contradiction.
(2) Suppose that $U = \{0\}$. Since $1 * 0 = 0 \in U$, there exist $V, W \in \tau$ such that $1 \in V, 0 \in W$ and $V * W \subseteq U$.

$$\{0, y\} = \{1 * 0, x * 0\} \subseteq F_1 * U \subseteq V * W \subseteq U,$$

which is a contradiction.

Case (4). If $F_1 = \{z, y, x, 1\}$, then $\tau_{F_1} = \tau$ by Lemma 5.2. □

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